

Hierarchical neighbor graphs: An energy-efficient bounded degree connected structure for wireless networks

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Abstract

We introduce hierarchical neighbor graphs, a new architecture for connecting ad hoc wireless nodes distributed in a plane. The structure has the flavor of hierarchical clustering and requires only local knowledge and minimal computation at each node to be formed and repaired. Hence, it is a suitable interconnection model for an ad hoc wireless sensor network. The structure is able to use energy efficiently by reorganizing dynamically when the battery power of heavily utilized nodes degrades and is able to achieve throughput, energy efficiency and network lifetimes that compare favorably with the leading proposals for data collation in sensor networks such as LEACH (Heinzelman et. al., 2002). Additionally, hierarchical neighbor graphs have low power stretch i.e. the power required to connect nodes through the network is a small factor higher than the power required to connect them directly. Our structure also compares favorably to mathematical structures proposed for connecting points in a plane e.g. nearest-neighbor graphs (Ballister et. al., 2005), θ -graphs (Ruppert and Seidel, 1991), in that it has expected constant degree and does not require any significant computation or global information to be formed.

1 Introduction

Topology control is a fundamental problem in the study of wireless ad hoc and sensor networks. A primary issue in this area is that of constructing a connected network between the nodes while keeping in mind the various constraints wireless devices operate under. Several architectures have been suggested to solve this problem, each striving to achieve multiple objectives vital to obtaining high throughput and low latency while expending as little energy as possible. Bounded degree is one such objective, needed to reduce the overhead of channel state exchange between neighbors when nodes use MIMO antennas [28]. Another important criterion of a good solution is that the number of hops between nodes be small, required because multi-hop wireless networks are error-prone and the probability of packet loss increases with path length. For wireless nodes with sufficient battery power, a connected topology must also ensure that the power stretch—the ratio of power spent by two nodes in connecting through the network to power spent in communicating directly—must be kept low [13, Chapter 9].

In this paper we propose a novel architecture for connectivity in wireless networks: Hierarchical neighbor graphs. Our structure is a randomized one that effectively combines ideas from skip list data structures [17, 2] and nearest-neighbor graphs [25, 3] to give a hierarchical bounded degree structure for connecting points in a plane that has short paths between nodes both in terms of number of hops and distance. Hierarchical neighbor graphs can be built with local information and without any substantial computation. In order to find connections, nodes do not need to estimate any angles or distances, they only need to know the relative distance of their neighbors, which can be easily determined from signal strength. Unlike other nearest-neighbor flavored structures, hierarchical neighbor graphs are able to incorporate battery power as a parameter and ensure that nodes with low battery power are not expected to transmit to nodes that are located far away. This property, along with ease of deployment and reformation as battery power decreases, makes our structure a good candidate for collating data from a field of wireless sensor nodes. And, in fact, we demonstrate that our structure is more energy efficient and has better throughput than one of the leading proposals in this area, LEACH [8], while having network lifetime comparable to it.

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The key idea of our construction is that each node is assigned a level and chooses one neighbor from a level above it and its other neighbors from its own level. This assignment of levels is done through a random process which helps ensure that the degree of each node is bounded in expectation no matter what the positions of the points. For the special case where the locations of the points are generated by a Poisson point process we show that even when all nodes have equal battery power, the probability of the edges formed being long is very small. Encouraged by this we define a edge-length bounded version of our structure in which all edges above a certain length, determined by the battery power of the node, are deleted. We demonstrate that for a Poisson point process of sufficiently high density, this more practical structure is still connected.

Paper organization and our contributions. In Section 2 we define hierarchical neighbor graphs. We also describe proactive and reactive routing protocols for this architecture and discuss how it can be easily adapted when nodes join or leave or when battery power changes. Additionally:

- We show analytically that hierarchical neighbor graphs have bounded degree in expectation (Section 3.1).
- When the location of the points are determined by a Poisson point process, the probability of long-range connections being formed is very low. We also define a bounded connection length version of our structure and show through simulation that it is connected for Poisson point processes of sufficiently high density (Section 3.2).
- We show analytically that the number of hops between any two nodes in our structure is low (Section 3.3).
- We show through simulation that the distance stretch of our structure has an exponentially decaying distribution, decaying faster for pairs of points that are further apart. We show an initial theorem in this direction (Section 3.4).
- We simulate the use of our architecture to collate data from a field of wireless sensors and show that it performs better than LEACH [8] in terms throughput and energy efficiency while matching it in terms of network lifetime (Section 4).

1.1 Related work

The area of topology control for wireless ad hoc and sensor networks has seen a lot of activity over the years. Several surveys and books are available on this topic (see e.g. [19, 21, 13]). The proposed architectures range from the geometry-based Gabriel Graphs [6] and Relative Neighborhood Graphs [22] and their numerous variants, and direction based proposals like Yao Graphs [26] and their family. Li gives a fairly comprehensive survey of these kinds of architectures in Chapter 9 of his book [13]. Of particular interest to us in this class of proposals are θ -graphs [20], in which the area around each node is partitioned into cones of angle θ and the node is connected to the nearest neighbor in each cone. These graphs can be shown to be spanners i.e. the graph distance between any two points is a bounded factor larger than their Euclidean distance. Arya, Mount and Smid [1] refined this proposal in a flavor very similar to ours by combining skip lists with θ -graphs to propose skip list spanners, which have the spanning property of θ -graphs but have fewer hops between all pairs on nodes— $\theta(\log n)$ hops for a point set with n nodes—than θ -graphs have. While Arya et. al.’s proposal seems very similar to ours, they have one major disadvantage they share with all the other structures of the family of geometric graphs: constructing these kinds of graphs normally involves computation involving the relative position of a point and its neighbors and an area surrounding these points. We note that our hierarchical neighbor graphs avoid the computation overhead that these kinds of construction require. This also makes them superior to the class of topology control proposals that are based on computing dominating sets or spanning trees (see e.g. [23]), computations which are time consuming and construct structures which are difficult to adapt in dynamic situations. This class of proposals contains several variants, too numerous to recount here, so we refer the reader to an excellent survey by Santi [21].

Clustering based approaches to topology control have also been studied extensively in the literature. Some of the common assumptions and algorithmic features of these proposals are location unawareness, periodic selection of cluster heads, equal battery power but multi-power transmission ability. The systems are evaluated by evaluating throughput and network lifetime under the assumption that battery power is not replenished. Within these constraints Younis and Fahmy proposed a distributed clustering approach, HEED, for maximizing lifetime of ad hoc sensor networks [27]. This approach was extended by Huang and Wu, with an additional assumption

of uniform distribution of nodes [10]. Basagni gave a method of choosing cluster heads based on a node-mobility parameter [4]. Heinzelman, Chandrakasan and Balakrishnan's architecture, LEACH, is an application specific protocol for ad hoc wireless sensors[8]. Cluster heads receive data from sensor nodes and compress it before sending to the base station. Similar to our approach, LEACH incorporates randomized rotation of high energy cluster head among sensors, keeping the expected number of cluster heads in the network constant. By contrast, in our approach the rotation is implicit and does not require additional computation. Also, for LEACH, nodes must have an estimate of energy remaining in the network at the end of each round. Another assumption LEACH makes that we do not share is that number of nodes in the network and the number of cluster heads must be known to every node. In Section 4 we compare our structure to LEACH and find that under similar assumptions regarding transmission energy, hierarchical neighbor graphs provide better throughput and energy efficiency with matching network lifetime. We note that systems like LEACH attempt to save power by randomly rotating cluster head responsibilities. A more sophisticated approach involves choosing nodes with greater residual energy to perform cluster head roles (See e.g. [24, 27]), and this is the paradigm in which hierarchical neighbor graphs fall.

Another method related to our approach in the sense that clustering is implicit is CLUSTERPOW introduced by Kawadia and Kumar [11]. No leader or gateway is explicitly selected rather the clustered structure of the network is manifested in the way routing is done. Each node can transmit at different finite number of power levels. A route is a non increasing sequence of power levels.

On the mathematical front, nearest neighbor models have been studied by Häggström and Meester [7] who proved that there is a critical value k_c dependent on the dimension of the space such that if each point is connected to its k nearest neighbors for any $k > k_c$, an infinite component exists almost surely. Restricting this model to a square box of area n , Xue and Kumar [25] showed that k must be at least $0.074 \log n$ for the graph to be connected. This lower bound was improved to $0.3043 \log n$ by Ballister, Bollobás, Sarkar and Walters [3] who also showed that the threshold for connectivity is at most $0.5139 \log n$ as well as corresponding results for the directed version of the problem. The advantage our model enjoys over this model is that by selecting neighbors carefully from the set of proximate nodes, hierarchical neighbor graphs achieve constant degree in expectation. Additionally our hierarchical structure also ensures paths with fewer hops.

Finally, we end by mentioning that hierarchical neighbor graphs bear more than a passing resemblance to the skip list data structure [17] and its biased version [2].

2 Hierarchical neighbor graphs

In this section we define hierarchical neighbor graphs and some special versions of them. We also describe routing protocols of these structures and methods for adapting them in dynamic settings. We will use the notation that for $u, v \in \mathbb{R}^2$, $d(u, v)$ is the Euclidean distance between the points.

Consider a set of points $V \subset \mathbb{R}^2$. We are given a function $w : V \rightarrow \mathbb{R}_+$ such that each node $u \in V$ has a battery power $c \cdot w(u)$ associated with it, where c is a constant determined by the minimum battery power a node needs to operate i.e. c ensures that any functioning node has weight at least 1. Taking a parameter p such that $0 < p < 1$, we form the p -hierarchical neighbor graph on V with weight function w , denoted $\text{HN}_p^w(V)$ as follows:

1. We create a sequence $\{S_n : n \geq 0\}$ of subsets of V such that $S_0 = V$. The point of a set S_{i-1} are "promoted" to S_i in two ways, one deterministic and one randomized.
 - Deterministically, all $u \in S_{i-1}$ with $\lfloor \log_{\frac{1}{p}} w(u) \rfloor \geq i$ are put into S_i .
 - The remaining points of S_{i-1} are placed in S_i with probability p independently of the choice of all other points.
2. After obtaining the sequence of sets, we say that the *level* of u , $\text{lev}_p(u) = i$ such that $u \in S_i$ and $u \notin S_{i+1}$.
3. Each point $u \in V$ grows a circle around it which stops growing the first time a point v with $\text{lev}(v) > i$ is encountered. This point is called *parent*(u). u makes connections to all nodes w with $\text{lev}(w) = i$ that lie within this circle and to the node(s) of S_{i+1} that lie on the circumference of the circle. Note that this algorithm is fully distributed in nature. All connections are made through local interactions.

Algorithm $\text{construct}(\text{HN}_p(V) = (V, \text{EN}_p))$

1. $\text{EN}_p = \phi$
2. At each node u do
 - (a) Determine $\text{lev}_p(u) = i$ such that $u \in S_i$ and $u \notin S_{i+1}$.
 - (b) Find the nearest neighbor $z \in S_{\text{lev}(u)+1}$ of u ;
 $\text{parent}(u) \leftarrow z$;
 $\text{EN}_p^{\text{lev}(u)} \leftarrow \text{EN}_p^{\text{lev}(u)} \cup \{(u, \text{parent}(u))\}$;
 - (c) For each $v \in S_{\text{lev}(u)} \setminus S_{\text{lev}(u)+1}$ such that $d(u, v) \leq d(u, \text{parent}(u))$ i.e v is closer to u than $\text{parent}(u)$
 $\text{EN}_p^{\text{lev}(u)} \leftarrow \text{EN}_p^{\text{lev}(u)} \cup \{(u, v)\}$;
3. $\text{EN}_p \leftarrow \text{EN}_p \cup \bigcup_{i=0}^{\infty} \text{EN}_p^i$.

Figure 1: Building subgraph $\text{HN}_p(V)$.

The algorithm for forming $\text{HN}_p^w(V)$ is described in Figure 1.

For the special case where $w(u) = 1$ for all $u \in V$, we denote the structure thus formed $\text{HN}_p(V)$.

2.0.1 Radius-bounded hierarchical neighbor graphs

We define a variant of hierarchical neighbor graphs, called radius-bounded hierarchical neighbor graphs, which take another function $r : V \rightarrow \mathbb{R}_+$ as a parameter. We denote this graph built on a point set V with weight function w and parameter p by $\overline{\text{HN}}_p^{r,w}(V)$. The function r limits the transmission radius of the nodes i.e. node u is not allowed to connect to any node v such that $d(u, v) > r(u)$. Hence $\overline{\text{HN}}_p^{w,r}(V)$ is a subgraph of $\text{HN}_p^w(V)$ and may not be connected. This model is, however, a more practical one, since it does not assume that nodes can form connections with distant neighbors. As before, we denote by $\overline{\text{HN}}_p^r(V)$, the special case of $\overline{\text{HN}}_p^{w,r}(V)$ where all weights are 1.

2.0.2 Repairing hierarchical neighbor graphs

Note that hierarchical neighbor graphs are adaptable in the sense that should a node u 's power decreases to $1/p$ times its original amount its level can be readjusted by reducing the deterministic part of its promotion by 1, without touching the probabilistic promotions. This leads to an overall decrease of 1 in $\text{lev}_p(u)$. Similarly if the power available with a node increases to $1/p$ times the original amount, the deterministic part of the promotion of a node can be increased by 1, thereby increasing $\text{lev}_p(u)$ by 1. Clearly when either of these happens the structure requires some repair which we discuss next

Node addition or battery power gain When a node u is added to V , it first determines $\text{lev}_p(u)$. Once its membership in sets $S_0, \dots, S_{\text{lev}_p(u)}$ is established, all points v with $\text{lev}_p(v) < \text{lev}_p(u)$ whose radius of connection contains u truncate their radius of connection and make u their parent. Nodes v with $\text{lev}_p(v) = \text{lev}_p(u)$ form connections to u if it lies within their radius of connection. All other nodes are unaffected. Similarly when a node u gains battery power, $\text{lev}_p^w(u)$ increases from i to j and nodes of S_i, S_{i+1}, \dots, S_j may have to truncate their radius of connection if u lies within it.

Node departure or battery power loss When a node u departs from V , all nodes that have u as their parent have to expand their radius of connection to find a new parent. Nodes in $S_{\text{lev}_p(u)}$ that were connected to u simply note its departure but don't have to change anything. Similarly when the battery power of a node u in $\text{HN}_p^w(V)$ decreases from i to j , nodes of $S_k \setminus S_{k+1}$ that had u as their parent have to increase their radius and find a new parent, for $j \leq k \leq i - 1$. Nodes of $S_j \setminus S_{j+1}$ simply note the departure of u from S_j .

2.1 Routing

We describe two approaches to routing on $\text{HN}_p(V)$, one proactive and one reactive. Before we do that, let us define some terms. The ancestor of u at level i , $\text{anc}_i(u)$, is the node of level at least i reachable from u through a series of edges connecting nodes to their parents. The *connected component of u at level i* , for $i \leq \text{lev}_p(u)$, denoted $\text{conn}_p(u, i)$ is the connected component containing u of the subgraph of $\text{HN}_p(V)$ comprising only of edges between vertices $v \in V$ such that $\text{lev}_p(v) \geq i$ i.e. it is the set of vertices whose are promoted at least up to level i and that are reachable from u without the path having to touch any node of highest level strictly less than i .

2.1.1 Proactive routing

In this method we extend the established ideas of distance vector based approaches. The algorithm proceeds in rounds, one for each level of the hierarchy, consolidating information from below and passing it up. The

Algorithm `hierarchicalDistanceVector($\text{HN}_p(V) = (V, \text{EN}_p)$)`

1. $i \leftarrow 0$
2. For all $u \in V, 0 \leq j \leq \text{lev}_p(u)$: $\text{directory}_p(u, j) \leftarrow \emptyset$.
3. While there is a node u with $\text{lev}(u) \geq i$ which is not marked visited
 - (a) Run a Distance Vector algorithm for $\text{conn}_p(u)$ in which each node v exchanges $\text{directory}(v, i)$ with all other nodes.
 - (b) For each $v \in \text{conn}_p(u)$ augment $\text{directory}_p(v, i)$ using the information gathered from the previous step and mark v visited.
 - (c) Send $\text{directory}_p(v, i)$ to $\text{parent}(v)$.
 - (d) Recompute $\text{directory}_p(\text{parent}(v), i)$ by adding the information in $\text{directory}_p(v, i)$.
 - (e) increment i

Figure 2: Constructing a routing table pro actively in $\text{HN}_p(V)$.

protocol is described in Figure 2. The idea is that at each level i each node v such that $\text{lev}_p(v) \geq i$ computes a distance vector to all the nodes in it's connected component. But along with the distances to a node v in its component, each node u also receives a distance vector directory of all nodes of levels lower than i that can be reached through v without visiting any node with highest level i or above. The node u first uses this information to update its distance vector since it is possible that a node with a highest level lower than i is better reached through some other node at level i rather than through links going down into the hierarchy below u . It then consolidates its distance vector information and sends it to $\text{parent}(u)$. The process then repeats at $\text{lev}(u)$.

When a packet needs to be sent from u to v , u discovers the best route by checking its directory at level 0. If it doesn't find v listed it queries $\text{parent}(u)$ which checks the directory at level 1 and so on till we reach a level i where $\text{anc}_i(u)$'s directory contains an entry for v . The route discovered first goes from u to $\text{anc}_i(u)$ and then proceeds to the node of S_i listed as the next hop for v in $\text{anc}_i(u)$'s directory.

2.1.2 Reactive routing

The problem with the proactive routing scheme is that it requires a lot of storage at each level and computation in the set up phase. There is also the problem of keeping the routing tables consistent with the current state of the network, which might be expensive in a situation which evolves quickly. For situations where storage may not be abundant, e.g. ad hoc sensor networks, and situations where nodes come and go quickly, e.g. mobile networks, we now propose a reactive routing scheme which uses a hierarchical controlled flooding method for route discovery.

The process of discovering a route from u to v proceeds in two kinds of phases, up the hierarchy and down the hierarchy. At first the source node u , floods $\text{conn}_p(u, 0)$ to find if v is in it. If it doesn't find it, it sends a message

to $\text{parent}(u)$. This node now floods $\text{conn}_p(\text{parent}(u), 1)$ asking for a route to v . Each node $w \in \text{conn}_p(\text{parent}(u), 1)$ sends messages down to all x s.t. $\text{parent}(x) = w$ asking them to flood their components at level 0 in search of v . Subsequently, if v is not found in $\text{conn}_p(\text{parent}(u), 1)$, a message is sent to $\text{parent}(\text{parent}(u))$ and it launches a flood at level 2, and so on. We do not give a formal description of the algorithm here, pointing instead to

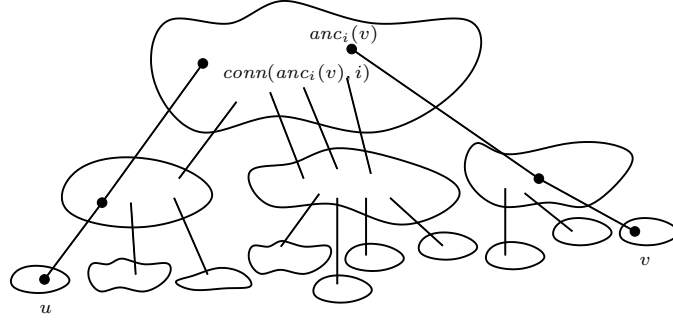


Figure 3: If there is an i such that $\text{anc}_i(u) \in \text{conn}(\text{anc}_i(v), i)$ then only the connected components lying at various levels beneath the vertices of $\text{conn}(\text{anc}_i(v), i)$ get flooded.

Figure 3 to give the reader an idea of the region of $\text{HN}_p(V)$ that gets flooded in the process of route discovery between u and v . We believe it is possible to argue that the number of nodes flooded in this process is $o(n)$ but we do not have a proof for this.

3 Properties of hierarchical neighbor graphs

3.1 Bounded degree

We first show that for *any* point set V the expected degree of the subgraph $\text{HN}_p(V)$ is $O(\frac{1}{p(1-p)})$. The proof of this can easily be adapted to show degree bounds for the case where weights are general, i.e. $\text{HN}_p^w(V)$. We also show that the degree bound for $\text{HN}_p(V)$ can be improved to $O(\frac{1}{p})$ when V is generated by a Poisson point process.

Theorem 3.1 *The expected degree of any point $v \in V$ in $\text{HN}_p(V)$ constructed with parameter p on an arbitrary point set P is at most*

$$\frac{1}{p} + \frac{6}{p(1-p)}.$$

Proof. Let us consider some point $v \in P$. For the purposes of analysis we think of the graph as directed and distinguish between *outgoing edges* i.e. the edges established by v and *incoming edges* i.e. the edges established to v by other points of P . We account for these two sets separately.

Outgoing edges. Let us assume that v is promoted up to level i . Clearly v establishes no outgoing edges till this level. At this level it stops being promoted and searches for the point nearest to it that is promoted to level $i + 1$. Let us name the points of S_i ordered by increasing distance from v as $u_1, u_2, \dots, u_{|S_i|}$. Clearly for v to have k outgoing edges, the first $k - 1$ vertices in this sequence must not be promoted past level i and the k th vertex must be promoted to level $i + 1$. Hence

$$P(v \text{ has } k \text{ outgoing edges}) = (1 - p)^{k-1} \cdot p.$$

Therefore

$$E(\text{No. of outgoing edges from } v) \leq \sum_{k=1}^{|S_i|} k \cdot (1 - p)^{k-1} \cdot p \leq \frac{1}{p}.$$

Incoming edges. Now let us consider a level $j \leq i$ i.e. a level at which v exists and may be connected to by other vertices of P . We denote by X_j the number of edges established into v by points that belong to $S_j \setminus S_{j+1}$ i.e. by

points that are promoted up to S_j but not beyond. Note that at level j the candidates to establish edges to v are looking for a vertex of level $j + 1$ to connect to. These vertices establish an edge with v either because none of the vertices nearer to them than v have been promoted to level $j + 1$.

In order to analyze this situation, let us consider the sequence $\sigma(v, j) = u_1, u_2, \dots, u_{|S_j|}$ of vertices of S_j ordered by their increasing distance from v . We partition the space around v into six cones subtending angles of $\frac{\pi}{3}$ at v and index these six partitions with the numbers 1 to 6. Further we partition the sequence $\sigma(v, j)$ into six subsequences $\sigma(v, j, i) = u_1^i, u_2^i, \dots, 1 \leq i \leq 6$ where the points of $\sigma(v, j, i)$ lie entirely within cone i . As is clear

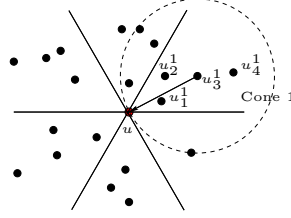


Figure 4: Note that u_3^1 cannot have an edge to v if either u_2^1 or u_1^1 have a higher level.

visually from Figure 4, using elementary geometry we can claim that the distance between two points within a cone is less than the distance between the further of these two points and v i.e.

Claim 3.2 *Given two points u_k^i and u_ℓ^i such that $k < \ell$*

$$d(u_k, u_\ell) < d(v, u_\ell).$$

Note that for v to have at least m incoming edges, at least $\frac{m}{6}$ of these edges must come from points of one of the six subsequences, say $\sigma(v, j, i^*)$. In view of Claim 3.2, this means that the first $k - 1$ points of $\sigma(v, j, i^*)$. Hence we can say that for v to have at least m incoming edges, the first $\lceil \frac{m}{6} \rceil$ points of $\sigma(v, j, i^*)$ must not be promoted to level $j + 1$ i.e.

$$P(v \text{ has at least } m \text{ incoming edges}) \leq (1 - p)^{\frac{m}{6}}. \quad (1)$$

Therefore $E(\text{No. of incoming edges to } v \text{ at level } j) \leq \sum_{m=1}^{|S_j|} (1 - p)^{\lceil \frac{m}{6} \rceil} \leq 6 \sum_{t=1}^{\lceil \frac{|S_j|}{6} \rceil} (1 - p)^t \leq \frac{6}{p}$.

What we have shown above is that $E(X_j) \leq \frac{6}{p}$, $j \geq 0$. Also, we know that for X_j to be non-zero, v must be promoted at least till level j , which happens with probability p^j . With this in view we calculate $E(\sum_{j \geq 0} X_j)$ i.e. the expectation of the total number of edges incoming to v . So, we have

$$\begin{aligned} E\left(\sum_{j \geq 0} X_j\right) &= \sum_{j \geq 0} p^j \cdot E(X_j) \\ &\leq \frac{6}{p(1 - p)}. \end{aligned}$$

□

The proof described above can easily be adapted to show the following theorem for $\text{HN}_p^w(V)$ for a general weight function:

Theorem 3.3 *The expected degree of any point $v \in V$ in $\text{HN}_p^w(V)$ constructed with parameter p on an arbitrary point set $V \subset \mathbb{R}^2$ is at most*

$$\frac{1}{p} + \frac{6}{p} \cdot \left(\log_{\frac{1}{p}} w(v) + \frac{1}{1 - p} \right).$$

We omit the proof here because it essentially follows the proof of Theorem 3.1, pointing out to the reader that inequality (1) holds as a pessimistic estimation of the probability in the weighted case, since the deterministic promotion of weighted nodes would make them higher in general than nodes of weight 1.

For point sets generated by a Poisson point process we show a better bound of $\theta(\frac{1}{p})$. We note that it is quite expected that the degree bound does not contain the intensity of the point process λ in it, since $\text{HN}_p^w(V)$ is essentially a nearest neighbor model.

Theorem 3.4 *The expectation of the degree of $\text{HN}_p(V)$ constructed with parameter p on a Poisson point process V with density λ is at most $\frac{7}{p}$.*

Proof. As before we account separately for outgoing edges and incoming edges. We inherit the bound for outgoing edges from the proof of Theorem 3.1 where the setting is more general. We focus here on improving the bound for incoming edges.

Consider a vertex v that has been promoted up to level i . Consider a level $j \leq i$. We denote by X_j the number of edges established into v by points that belong to $S_j \setminus S_{j+1}$ i.e. by points that are promoted up to S_j but not beyond.

We partition the space around v into six cones subtending angles $\frac{\pi}{3}$ at v and number them from 1 to 6. Consider a point u at a distance r from v in cone k ; $1 \leq k \leq 6$. Let X_j^k denotes the edges from vertices in cone k . By the symmetry of the cones and linearity of expectation we can say that

$$E(X) = 6 \cdot E(X_1).$$

So, we focus on one of these cones and compute an upper bound on the expected number of edges incoming to v from this cone. In view of Claim 3.2, for a point $u \in S_j \setminus S_{j+1}$ to connect to v all the points of S_j in cone 1 which are nearer to v than u must not be promoted to level $j+1$ i.e. they must all belong to $S_j \setminus S_{j+1}$. Note that this is an upper bound since we disregard the points outside the cone which might be closer to u than v which might be promoted to S_{j+1} and also the points within the cone but further from v than u which might be closer to u than v is and which might be promoted to S_{j+1} . Both these kinds of points would prevent u from sending an edge into v but we disregard them since we are only looking for an upper bound.

In order to compute this upper bound on $E(X_j^1)$ we consider a segment of the cone at a distance r from v which infinitesimal width dr . Since the area of this strip is infinitesimal, the probability that there are 2 or more points in this strip is $o(dr)$ and hence can be neglected. Hence the expected number of edges from this strip into v can be upper bounded by the probability that there is an edge from within this strip going into v . This is computed by computing the intersection of the events “the $i+1$ st nearest neighbor of v in cone 1 is in the strip of width dr at distance r ” and “none of the i points of S_j in the sector of radius r in cone 1 are promoted to level $j+1$ ”. Since these two events happen on disjoint areas they are independent. To compute the expectation of X_j^1 we simply integrate over all values of r from 0 to ∞ for each value of i and sum over all values of i from 0 to ∞ i.e.

$$\begin{aligned} E(X_j^1) &\leq (1-p) \cdot \int_0^\infty \sum_{i=0}^\infty \frac{e^{-\lambda p^j \frac{\pi}{6} r^2} \cdot (\lambda p^j \frac{\pi}{6} r^2)^i (1-p)^i}{i!} \\ &\quad \cdot \lambda p^j \frac{\pi}{3} r dr \\ &= (1-p) \cdot \int_0^\infty e^{-\lambda p^j \frac{\pi}{6} r^2} \cdot \lambda p^j \frac{\pi}{3} r dr \\ &= \frac{1-p}{p} \end{aligned}$$

So we get that

$$\begin{aligned} E\left(\sum_{j \geq 0} X_j\right) &\leq 6 \sum_{j \geq 0} p^j \cdot E(X_j^1) \\ &\leq 6 \cdot \frac{1}{1-p} \cdot \frac{1-p}{p} \\ &= \frac{6}{p}. \end{aligned}$$

□

As before, there is a theorem analogous to Theorem 3.4 for the case where weights are general. We state that theorem here, noting that we do not use the terminology of a Point process in \mathbb{R}^2 since that would have, in general, an infinite number of points and if the weight of each of these points were considered to be at least some $\epsilon > 0$, that would amount to an assumption of infinite energy.

Theorem 3.5 *The expected degree of any point $v \in V$ in $\text{HN}_p^w(V)$ constructed with parameter p on a finite set of points V distributed uniformly at random in a bounded area $A \subset \mathbb{R}^2$ is at most*

$$\frac{1}{p} + \frac{6}{p} \cdot \left(\log_{\frac{1}{p}} w(v) + 1 \right).$$

The proof of this theorem follows the proof of Theorem 3.3 and is omitted here.

Discussion The result of Theorem 3.4 assumes significance when seen in light of Ballister et. al.'s result on connectivity of k -nearest neighbor graphs [3]. They considered k -nearest neighbor graphs on point sets in \mathbb{R}^2 i.e. each node establishes edges with its k nearest neighbors. Improving on a result of Xue and Kumar [25], Ballister et. al. showed that for a Poisson point process with $\lambda = 1$ on a square of area n , the probability of the k -nearest neighbor graph being connected tends to 0 as $n \rightarrow \infty$ for $k \leq \lfloor 0.3043 \log n \rfloor$. In contrast, our p -hierarchical neighbor graphs achieve connectivity, even within a square of area n with an expected degree not depending on n .

Also, we note that nodes in structures like θ -graphs [12, 20] and Yao [26] graphs have constant outgoing degree but may have arbitrarily high incoming degree. In fact several papers have been devoted to constructing constant degree versions of these structures at some computational expense (see [13, Chapter 9] for a thorough treatment). Theorem 3.1 does not preclude the possibility of a node having high degree in hierarchical neighbor graphs, the bound on degree is only in expectation, but it has the advantage that the kind of positioning of points that prove pathological for θ -graph-like structures, still retain some probability of having bounded node degree, at no extra computation cost.

3.2 Expected edge length in $\text{HN}_p(V)$

Since nodes in wireless networks are driven by (limited) power batteries, their transmission ranges are finite. A network architecture ignoring this limitation is impractical. In order to show that hierarchical neighbor graphs are sensitive to this constraint we will show that the probability of long connections being formed in $\text{HN}_p(V)$ is very low when V is generated by a Poisson point process of density λ . As a consequence of this we expect that for every value of $r > 0$, the radius-bounded hierarchical neighbor graph $\overline{\text{HN}}_p^{w,r}(V)$ is connected as long as λ is sufficiently high. We demonstrate by simulation that this is indeed the case. In fact, for every $r > 0$ there is a value $\lambda_{\min}(r)$ such that $\overline{\text{HN}}_p^{I,r}(V)$ is connected for all $\lambda > \lambda_{\min}(r)$, for the special weight function I that assigns weight 1 to all nodes.

We begin by bounding the probability that 2 nodes which are distance l apart are connected directly.

Proposition 3.6 *Consider $\text{HN}_p(V)$ constructed on a point set V generated by a Poisson point process of density λ . Given any two nodes $u, v \in V$, with $d(u, v) = l$, the probability that u forms an edge with v or vice versa in $\text{HN}_p(V)$, denoted $g_{\leftrightarrow}(l)$, is upper bounded as*

$$g_{\leftrightarrow}(l) \leq \frac{2(1-p)}{(\lambda\pi l^2 p)^2} \cdot \left\{ \frac{1 - e^{-\lambda\pi l^2 p}(\lambda\pi l^2 p + 1)}{\log(1/p)} + \frac{4}{e^2} \right\}.$$

Proof. We use $g_{\leftrightarrow}(l)$ to represent this probability. Consider 2 nodes u and v s.t. $\text{lev}_p(u) = i$, $\text{lev}_p(v) = j$ and $d(u, v) = l$. Note that if $i < j$, u connects to v iff there is no node w s.t. $\text{lev}_p(w) > i$ and $d(u, w) < d(u, v)$. If

$i = j$, u and v have an edge iff there is no node w s.t. $lev_p(w) > i$ and $d(u, w) < d(u, v)$ or $d(w, v) < d(u, v)$.

$$\begin{aligned}
g_{\leftrightarrow}(l) &\leq \sum_{i=0}^{\infty} \sum_{j \neq i} p^i (1-p) p^j (1-p) e^{-\lambda \pi l^2 p^{\min\{i,j\}+1}} \\
&\quad + \sum_{i=0}^{\infty} 2p^{2i} (1-p)^2 e^{-\lambda \pi l^2 p^{i+1}} \\
&= 2(1-p) \cdot \left(\sum_{i=0}^{\infty} p^{2i} e^{-\lambda \pi l^2 p^{i+1}} \right)
\end{aligned}$$

Now we upper bound the summation by integrating over the function $p^{2x} e^{-\lambda \pi l^2 p^{x+1}}$, giving us the result. \square

In the radius-bounded hierarchical neighbor graph, $\overline{\text{HN}}_p^{w,r}(V)$ we go a step further towards incorporating real-life constraints and remove all connections between nodes more than a certain distance apart. Our simulations showed that for $\overline{\text{HN}}_p^r(V)$ built on a uniformly and randomly distributed points is still connected for all $r > 0$, provided the density of these points exceeds a minimum value that depends on r . To investigate the relationship between this minimum density (λ_{\min}) and transmission radius (r), we simulated $\overline{\text{HN}}_p^r(V)$ increasing the density of the points till we achieved a connected network. In order to verify that the value we determined as λ_{\min} was not an anomaly, we ran the simulation for 10 increments of λ past the first value where a connected network was achieved, and only fixed λ_{\min} when the network was found to be connected for all these 10 increments. Figure 5

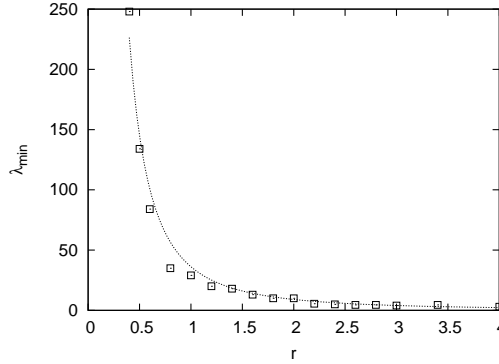


Figure 5: For $\overline{\text{HN}}_{0.5}^r(V)$ the curve is $\lambda_{\min} r^2 = 3.62$. (Best fit)

shows the dependency of λ_{\min} on r . We fixed $p = 0.5$. The point set was scattered randomly within a 10×10 square. On the Y-axis is λ_{\min} (above which network is connected) plotted against r , the max transmission radius allowed. We found the relation to be of the form $\lambda_{\min} \cdot r^2 = c$, c being a constant. This is reminiscent of the critical phenomena of unit disk graphs built on Poisson point processes. If the radius of each disk is r , there is a critical density $\lambda_c(r)$ above which the unit disk graph has an infinite component. And it can be shown that $\lambda_c(r) \cdot r^2$ is a constant for $r > 0$ (See [15] for details). We feel such a theorem may exist for $\overline{\text{HN}}_p^r(V)$.

3.3 Number of hops

To bound communication delay in networks, one must construct a topology with a small hop-stretch factor. Hop spanners were introduced by Peleg and Ullman [16] and were used as network synchronizers. In this section we analyze the hop stretch in hierarchical neighbor graphs.

For finite point sets and Poisson point processes limited to finite regions it is easy to show exponential decay of the number of hops between points. We introduce the notation $ht(\text{HN}_p^W(V)) = \max_{u \in V} lev_p(u)$. For finite point sets, we claim the following theorem, that follows easily from standard skip list analysis.

Theorem 3.7 For $\text{HN}_p^w(V)$ constructed on a finite set V , define $W(V) = \sum_{u \in V} w(u)$. For $k \geq \max_{u \in V} \log_{\frac{1}{p}} w(u)$,

$$P(\text{ht}(\text{HN}_p^w(V)) \geq k) \leq W \cdot p^k.$$

Proof. For a given node $u \in V$, the probability that $\text{lev}_p(u) > k$, for any $k \geq \max_{u \in V} \lceil \log_{\frac{1}{p}} w(u) \rceil$ is $p^{k - \log_{\frac{1}{p}} w(u)}$. And since the height of $\text{HN}_p^w(V)$ is at least k if there is at least one node with height at least k , applying the union bound on probabilities we get that $P(\text{ht}(\text{HN}_p^w(V)) \geq k) \leq \sum_{u \in V} p^{k - \log_{\frac{1}{p}} w(u)} = W \cdot p^k$. \square

The nodes at level $\text{ht}(\text{HN}_p^w(V))$ are fully connected to each other (since they are not able to find a node at a higher level). This means that in $\text{HN}_p(V)$ on a finite set V , for any two nodes to connect to each other the path has to travel at most $O(\text{ht}(\text{HN}_p^w(V)))$ levels up and down the hierarchy. Hence Theorem 3.7 implies that the number of hops between any two nodes has an exponentially decaying distribution.

For point processes limited to finite regions, the following theorem gives us the result that the number of hops between points decays exponentially:

Theorem 3.8 Given a finite region $A \subseteq \mathbb{R}^2$, with area $\ell(A)$, and a set of points V in this region generated by a Poisson point process of density λ , then for any $k \geq 0$,

$$P(\text{ht}(\text{HN}_p(V)) \geq k) \leq \lambda \cdot \ell(A) \cdot p^k.$$

The proof follows easily by conditioning on the number of points in A and using the argument from the proof of Theorem 3.7 to bound the conditional probability of the height being greater than k . We omit the details. Theorem 3.8 can be extended to general weights as well, but we would have to carefully define how the weights are distributed by the Poisson point process. We omit that case here since it is basically a mathematical digression.

For Poisson point processes in \mathbb{R}^2 we found through simulation that the number of hops varies logarithmically with the distance between the pair of points being connected. We do not have an analytical proof for this fact as yet.

3.4 Bounding the stretch

A major concern of topology control mechanisms is that the graph be a spanner i.e. given a point set V and a interconnection structure G , if we denote the shortest distance between points $u, v \in V$ along the edges of G by $d_G(u, v)$, the ratio

$$\delta_G = \max_{u, v \in V} \frac{d_G(u, v)}{d(u, v)},$$

known as the *distance stretch* of G should be low. For example θ -graphs have distance stretch $\frac{1}{1 - \sin(\theta/2)}$ [20]. The *power stretch* of G is defined as the ratio of the power expended by communicating through the links of G to the power expended in communicating directly. The power stretch of G is known to be upper bounded by δ_G^β [14], where β is a constant between 2 and 5 that depends on the medium, hence we only consider distance stretch here.

In order to study the spanner properties of hierarchical neighbor graphs, we performed a simulation and computed the distance stretch of $\text{HN}_p(V)$ constructed on a set of points randomly distributed on a torus of unit area. In Figure 6 we plot the graph distance of pairs of points of V that have a Euclidean distance of 0.1. We found that the log of the number of pairs of vertices stretched to extent s decreases linearly with s which leads us to believe that the probability of a pair of vertices being stretched decays exponentially with the distance stretch value. We also found that the number of pairs with lower stretch values dominates the number of pairs with higher stretch values as the distance between the pairs increases. These observations lead us to the conjecture that the stretch for pairs of points in $\text{HN}_p(V)$ follows a probability distribution that decays exponentially in the stretch value, and also decreases as the distance between the pairs increases, which effectively means that distant points undergo very little distortion in their distance when connected through $\text{HN}_p(V)$, although nearby points may occasionally have to connect to each other through a long route.

Analytically, we were able to prove an initial theorem that confirms our simulation results, but gives a weaker result.

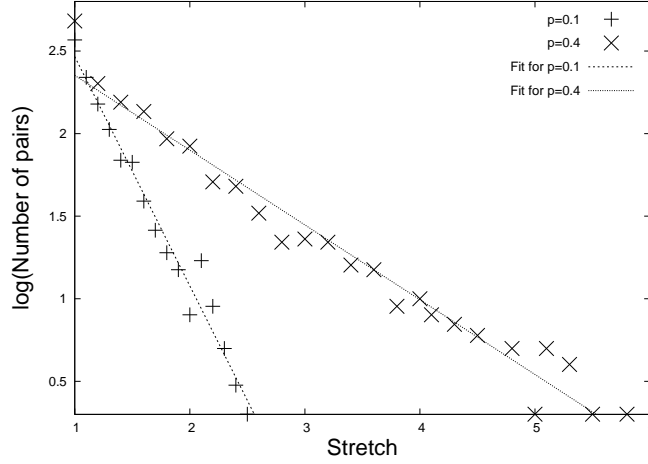


Figure 6: Stretch graphs for pairs with $d(u, v) = 0.1$. $\lambda = 500$.

Theorem 3.9 *Given parameters p and λ s.t. $0 < p < 1$ and $\lambda > 0$, the graph $\text{HN}_p(V)$ built on a set of points V generated by a Poisson point process with intensity λ has the property that for any two points $u, v \in V$ such that $d(u, v) = l$, there are positive constants c_1 and $c_2 < 1$ depending only on p and λ such that for $0 < \theta < \frac{\pi}{3}$,*

$$P\left(d_p(u, v) > \frac{l}{1 - 2 \sin\left(\frac{\theta}{2}\right)}\right) \leq \exp\left\{-\frac{c_1}{l^4 \cdot (1 - c_2 \cdot \frac{\theta}{\pi})}\right\}.$$

Before proceeding with the proof of Theorem 3.9, we need some preliminaries. Given a point set V , our algorithm for constructing $\text{HN}_p(V)$ defines a probability space (Ω, \mathcal{F}, P) . A realization of this process i.e. an element of Ω , is of the form $\omega = (V = S_0, S_1, \dots)$. Let us denote the i th set in this tuple as $\omega(i)$. We define a partial order on the set of all realizations as follows, $\omega \preceq \omega'$ if $\omega(i) \subseteq \omega'(i)$ for all $i \geq 0$. An random variable N defined on this probability space is called an *increasing* random variable if $N(\omega) \leq N(\omega')$ whenever $\omega \preceq \omega'$. Similarly a random variable N is called decreasing if $-N$ is increasing. An event is called increasing if its indicator is an increasing random variable. For our setting this definition allows us to state a specific version of the more general FKG inequality [5].

Lemma 3.10 *For any two events A and B that are either both increasing or both decreasing,*

$$P(A \cap B) \geq P(A) \cdot P(B),$$

where $P(\cdot)$ is the probability measure defined on Ω .

Now we turn to the main proof.

Proof. In order to prove this theorem we draw on the paper on θ -graphs by Ruppert and Seidel [20] in which they extended earlier work by Keil [12] and showed that paths that stayed within a cone of angle θ centered around the line joining a point to its destination have stretch at most $1/(1 - 2 \sin(\theta/2))$. The proof proceeds by calculating a lower bound on the probability that there is a path between any two points that stays within a cone of angular width θ .

Given two points u and v , we define the θ -cone of u w.r.t v as a sector of a circle of radius $d(u, v)$ centered at u with an angle θ which is bisected by the line segment joining u and v . We denote this cone $\nabla_v^\theta(u)$. We define a θ -good path between vertices u and v as a path $u = u_0 \rightarrow u_1 \dots \rightarrow u_n = v$ such that u_{i+1} is in $\nabla_v^\theta(u_i)$.

We also define events that will be relevant to us

- G_{uv}^θ : There is a θ -good path between vertices u and v .
- $G_{uv}^\theta(i, j)$: There is a θ -good path between vertices u and v conditioned on the fact that $\text{lev}(u) = i$ and $\text{lev}(v) = j$.

- $G_{u \leftrightarrow v}$: The event that there is a direct edge between vertices u and v .
- $G_{u \leftrightarrow v}(i, j)$: The event that there is a direct edge between vertices u and v conditioned on the fact that $\text{lev}(u) = i$ and $\text{lev}(v) = j$.

Our proof proceeds by computing the probability that the two points are either directly connected or u is connected to a point w in $\mathcal{V}_v^\theta(u)$ which then has a θ -good path joining it to v . Note that the second case is an intersection event. Hence in order to proceed we will need the following lemma which shows that the two events being intersected are positively correlated.

Lemma 3.11 *Given a set of points X generated by a Poisson point process in \mathbb{R}^2 , consider three points $u, v, w \in X$ such that $w \in \mathcal{V}_v^\theta(u)$. Then*

$$P\left(\bigcup_{k=0}^{\infty} \{G_{u \leftrightarrow w}(i, k) \cap G_{wv}^\theta(k, j)\}\right) \geq \sum_{k=0}^{\infty} P(G_{u \leftrightarrow w}(i, k)) \cdot P(G_{wv}^\theta(k, j)).$$

Proof. We proceed by showing that for a given realization of the Poisson point process the events we are considering are both decreasing. Hence we can apply the FKG inequality to these events conditioned on a particular realization, then decondition to get the result. Let us formalize this.

Following [15] we view the Poisson point process C in the following way: Consider the family of boxes

$$K(n, z = (z_1, z_2)) = \left(\frac{z_1}{2^n}, \frac{z_1 + 1}{2^n}\right] \times \left(\frac{z_2}{2^n}, \frac{z_2 + 1}{2^n}\right], \text{ for all } n \in \mathbb{N}, z \in \mathbb{Z}^2,$$

For a given realization of X , each point $x \in X$ is contained in a unique box $K(n, z(n, x))$ for each $n \in \mathbb{N}$. With probability 1 there is a unique smallest number $n(x)$ such that $K(n(x), z(n(x), x))$ contains no other points of X . Hence we can view the realizations of X in terms of sets of boxes taken from

$$\mathcal{K}(\mathbb{Z}^2) = \bigcup_{n \in \mathbb{N}} \bigcup_{z \in \mathbb{Z}^2} K(n, z),$$

such that each box contains exactly one point i.e. we have a measurable mapping from the probability space of the Poisson point process to a space $\Omega_1 \subseteq \mathcal{K}(\mathbb{Z}^2)$ and an outcome $\omega \in \Omega_1$ is a set of boxes containing exactly one point each.

In order to describe the probability space on which $\text{HN}_p(X)$ is defined, we observe that each realization of $\text{HN}_p(X)$ can be completely described by the levels of the points of a particular realization of X . Defining $\Omega_2 = \mathbb{N} \cup \{0\}$ we get that $\text{HN}_p(X)$ is defined on a product space $\Omega = \Omega_1 \times \Omega_2$.

Now, let us consider a given $\omega \in \Omega_1$ and two points $u, v \in X$ for this outcome. Let us denote by $G(\omega)_{u \leftrightarrow v}(i, j)$ the event that for the realizations mapped to ω , u and v are connected by a directed edge, conditioned on the fact that $\text{lev}(u) = i$ and $\text{lev}(v) = j$. We claim that this is a decreasing event. To see this note that the event $G(\omega)_{u \leftrightarrow v}(i, j)$ depends only on the levels of all the points of ω apart from u and v . Consider two outcomes α and α' contained in the subspace of Ω defined by ω_1 and $\{\text{lev}(u) = i, \text{lev}(v) = j\}$. Assume that $\alpha \preceq \alpha'$ as defined at the beginning of this section.

Let us assume that $i \leq j$. Clearly if $G(\omega)_{u \leftrightarrow v}(i, j)$ does not occur in α , there is some point $w \in \omega_1$ such that $\text{lev}(w)(\alpha)$ i.e. the value of $\text{lev}(w)$ in outcome α , is strictly greater than i and $d(u, w) \leq d(u, v)$. Since $\alpha \preceq \alpha'$, this point w exists in α' and $\text{lev}(w)(\alpha') \geq \text{lev}(w)(\alpha) > i$ and hence there cannot be a direct edge from u to v in α' either. If $i = j$, we can find two such points that prevent a direct edge being formed between u and v and exactly the same argument holds. We state this as a proposition.

Proposition 3.12 *Given an $\omega \in \Omega_1$ and two points u, v of the Poisson point process, $G(\omega)_{u \leftrightarrow v}(i, j)$ is a decreasing event for all integers $i, j \geq 0$.*

Again starting with an $\omega \in \Omega_1$ and a sequence of points $\mathbf{x} = x_1, \dots, x_n, n \geq 2$ of a realization of the Poisson point process mapped to ω , we define the event

$$A(\omega)\mathbf{x} : G_{x_1 \leftrightarrow x_2} \cap \dots \cap G_{x_{n-1} \leftrightarrow x_n},$$

and given another sequence $\mathbf{i} = i_1, \dots, i_n, n \geq 2$ of non-negative integers, we define

$$A(\omega)\mathbf{x}(\mathbf{i}) = \{A(\omega)\mathbf{x} \mid \text{lev}(x_j) = i_j, 1 \leq j \leq n\}.$$

In other words, the event $A(\omega)\mathbf{x}(\mathbf{i})$ is the event that the points of \mathbf{x} form a path in $\text{HN}_p(X)$ conditioned on their levels being fixed. We claim that events of the form $A(\omega)\mathbf{x}(\mathbf{i})$ are decreasing events.

Proposition 3.13 *Given an $\omega \in \Omega_1$ and a sequence of points $\mathbf{x} = x_1, \dots, x_n, n \geq 2$ of the point process, $A(\omega)\mathbf{x}(\mathbf{i})$ is a decreasing event for every sequence $\mathbf{i} = i_1, \dots, i_n, n \geq 2$ of non-negative integers.*

This is not difficult to see since the event $A(\omega)\mathbf{x}(\mathbf{i})$ is a finite intersection of the kind of events we argued were decreasing in Proposition 3.12.

Before proceeding we observe that a bounded area around $\vee_v^\theta(u)$ is all that we are considering since points outside a certain region do not affect the events we are discussing. In the following when we talk about an outcome $\omega \in \Omega_1$, we will in fact only be referring to that outcome limited within this bounded region.

Now we turn to the left hand side of the inequality in the statement of the lemma and note that it can be rewritten as

$$\sum_{\omega \in \Omega_1} \text{P} \left(\bigcup_{k=0}^{\infty} \{G(\omega)_{u \leftrightarrow w}(i, k) \cap G(\omega)_{wv}^\theta(k, j)\} \right) \cdot \text{P}(\omega), \quad (2)$$

where $\text{P}(\omega)$ is the probability of a particular outcome. Note that since we have restricted our outcomes to a finite region containing $\vee_v^\theta(u)$, the probability $\text{P}(\omega)$ is non-zero.

We observe that the event $G(\omega)_{wv}^\theta(k, j)$ can be written as a union of events of the form $A(\omega)\mathbf{x}(\mathbf{i})$ where $x_1 = w$ and $x_n = v$ as long as all the points of \mathbf{x} have the property of θ -good paths. To write this formally we define $\sigma^n(\omega)(w, v)$ to be all the sequences \mathbf{x} of points that have these properties i.e. $x_1 = w, x_n = v, x_i \in \vee_v^\theta(x_{i-1}), 2 \leq i \leq n$. Also we use the notation $\mathbb{N}_0^i = \prod_{j=1}^i \mathbb{N} \cup \{0\}$. Hence we get

$$G(\omega)_{wv}^\theta(k, j) = \bigcup_{n=2}^{\infty} \bigcup_{\mathbf{i} \in \mathbb{N}_0^{n-2}} \bigcup_{\mathbf{x} \in \sigma^n(\omega)(w, v)} A(\omega)\mathbf{x}(\mathbf{i}).$$

Putting this into (2) we get that

$$\begin{aligned} \text{LHS} &= \sum_{\omega \in \Omega_1} \text{P} \left(\bigcup_{k=0}^{\infty} \bigcup_{n=2}^{\infty} \bigcup_{\mathbf{i} \in \mathbb{N}_0^{n-2}} \bigcup_{\mathbf{x} \in \sigma^n(\omega)(w, v)} \{G(\omega)_{u \leftrightarrow w}(i, k) \cap A(\omega)\mathbf{x}(\mathbf{i})\} \right) \cdot \text{P}(\omega) \\ &= \sum_{\omega \in \Omega_1} \sum_{k=0}^{\infty} \sum_{n=2}^{\infty} \sum_{\mathbf{i} \in \mathbb{N}_0^{n-2}} \sum_{\mathbf{x} \in \sigma^n(\omega)(w, v)} \text{P}(\{G(\omega)_{u \leftrightarrow w}(i, k) \cap A(\omega)\mathbf{x}(\mathbf{i})\}) \\ &\quad \cdot \text{P}(\omega) \cdot \text{P}(\text{lev}(w) = k) \cdot \text{P}(\text{lev}(x_j) = i_j, 2 \leq j \leq n-1). \end{aligned}$$

From Propositions 3.12 and 3.13 and the FKG inequality (Lemma 3.10) we get that

$$\begin{aligned} \text{LHS} &\geq \sum_{\omega \in \Omega_1} \sum_{k=0}^{\infty} \sum_{n=2}^{\infty} \sum_{\mathbf{i} \in \mathbb{N}_0^{n-2}} \sum_{\mathbf{x} \in \sigma^n(\omega)(w, v)} \text{P}(G(\omega)_{u \leftrightarrow w}(i, k)) \cdot \text{P}(A(\omega)\mathbf{x}(\mathbf{i})) \\ &\quad \cdot \text{P}(\omega) \cdot \text{P}(\text{lev}(w) = k) \cdot \text{P}(\text{lev}(x_j) = i_j, 2 \leq j \leq n-1) \\ &\geq \sum_{\omega \in \Omega_1} \sum_{k=0}^{\infty} \text{P}(G(\omega)_{u \leftrightarrow w}(i, k)) \cdot \text{P}(\text{lev}(w) = k) \cdot \text{P}(\omega) \\ &\quad \cdot \sum_{\omega \in \Omega_1} \sum_{n=2}^{\infty} \sum_{\mathbf{i} \in \mathbb{N}_0^{n-2}} \sum_{\mathbf{x} \in \sigma^n(\omega)(w, v)} \text{P}(A(\omega)\mathbf{x}(\mathbf{i})) \cdot \text{P}(\omega) \cdot \text{P}(\text{lev}(x_j) = i_j, 2 \leq j \leq n-1) \\ &= \text{RHS} \end{aligned}$$

□

Since the underlying point set V is produced by a stationary process, the probability of having a particular path between a pair of points a certain distance apart is the same as the probability of having the same kind of path for another pair which is the same distance apart. Hence we define the following functions which we will need

- $g(l) = P(G_{uv}^\theta \mid d(u, v) = l)$.
- $g(i, j, l) = P(G_{uv}^\theta \mid d(u, v) = l, \text{lev}_p(u) = i, \text{lev}_p(v) = j)$.
- $g_{\leftrightarrow}(l) = P(G_{u \leftrightarrow v} \mid d(u, v) = l)$.
- $g_{\leftrightarrow}(i, j, l) = P(G_{u \leftrightarrow v} \mid \text{lev}_p(u) = i, \text{lev}_p(v) = j, d(u, v) = l)$.

We start by looking at $g(i, j, l)$ for some $i, j > 0$. We divide this probability into two cases. The first being that there is a direct edge between u and v , and the second that there is a direct edge between u and a point w of level k which further has a θ -good path to v . We sum this probability over all possible values of k . Note that when $i \leq k$, u connects to w when looking for a point of level $i + 1$, and when $i \geq k$, w connects to u when looking for a point of level $k + 1$. In case w gives an incoming edge to u , we sum over the probability that the closest neighbor of u belonging to level k has a θ -good path to v . We get that

$$\begin{aligned}
P(G_{uv}^\theta(i, j)) &= P(G_{u \leftrightarrow v}(i, j)) + (1 - P(G_{u \leftrightarrow v}(i, j))) \cdot P\left(\bigcup_{w \in \mathcal{V}_v^\theta(u)} G_{u \leftrightarrow w} \cap G_{wv}^\theta\right) \\
&= P(G_{u \leftrightarrow v}(i, j)) + (1 - P(G_{u \leftrightarrow v})) \cdot P\left(\bigcup_{w \in \mathcal{V}_v^\theta(u)} \bigcup_{k \geq 0} G_{uw}(i, k) \cap G_{wv}^\theta\right) \\
&= P(G_{u \leftrightarrow v}) + (1 - P(G_{u \leftrightarrow v})) \\
&\quad \cdot \int_0^{d(u, v)} P\left(\bigcup_{w \in \mathcal{V}_v^\theta(u)} \bigcup_{k \geq 0} \{G_{uw}(i, k) \cap G_{wv}^\theta \mid x \leq d(u, w) \leq x + dx\}\right) \\
&\quad \cdot P(\exists w \in \mathcal{V}_v^\theta(u), x \leq d(u, w) \leq x + dx).
\end{aligned}$$

Now since we are considering infinitesimal widths, the probability that there is more than one point at a distance between x and $x + dx$ from u in $\mathcal{V}_v^\theta(u)$ is $o(dx)$ so we can neglect it. Hence we get

$$P(G_{uv}^\theta(i, j)) = \int_0^{d(u, v)} P\left(\bigcup_{k \geq 0} \{G_{uw}(i, k) \cap G_{wv}^\theta \mid x \leq d(u, w) \leq x + dx\}\right) \cdot \lambda \theta x dx.$$

Now, using the result of Lemma 3.11 we get

$$\begin{aligned}
P(G_{uv}^\theta(i, j)) &\geq \sum_{k=0}^{\infty} \int_0^{d(u, v)} P(G_{u \leftrightarrow w}(i, k) \mid x \leq d(u, w) \leq x + dx) \\
&\quad \cdot P(G_{wv}^\theta(k, j) \mid x \leq d(u, w) \leq x + dx) \cdot \lambda \theta x dx.
\end{aligned}$$

For $k \leq i$ we have

$$P(G_{uv}^\theta(i, j)) \geq \sum_{k=0}^{\infty} \int_0^{d(u, v)} e^{-\lambda \pi x^2 p^k} \cdot e^{-\lambda \pi x^2 p^{k+1}} \lambda \theta x dx.$$

and for $k > i$

$$P(G_{uv}^\theta(i, j)) \geq \sum_{k=0}^{\infty} \int_0^{d(u, v)} e^{-\lambda \pi x^2 p^k} \cdot e^{-\lambda \pi x^2 p^{i+1}} \lambda \theta x dx.$$

Let us assume that $d(u, v) = l$ In the previous equation if $d(w, v) = \delta$ we get

$$\begin{aligned} g(i, j, l) &\geq g_{\leftrightarrow}(i, j, l) \\ &+ (1 - g_{\leftrightarrow}(i, j, l)) \cdot \sum_{k=0}^i \left(\int_0^l e^{-\lambda \pi x^2 p^k} \cdot e^{-\lambda \pi x^2 p^{k+1}} \lambda \theta x dx \right) \cdot p^k (1-p) \cdot g(k, j, l - \delta) \\ &+ (1 - g_{\leftrightarrow}(i, j, l)) \cdot \sum_{k=i+1}^{\infty} \left(\int_0^l e^{-\lambda \pi x^2 p^k} \cdot e^{-\lambda \pi x^2 p^{i+1}} \lambda \theta x dx \right) \cdot p^k (1-p) \cdot g(k, j, l - \delta). \end{aligned}$$

For $n > m$

$$e^{-\lambda \pi x^2 p^n} > e^{-\lambda \pi x^2 p^m}.$$

Putting these values in we get

$$\begin{aligned} g(i, j, l) &\geq g_{\leftrightarrow}(i, j, l) + (1 - g_{\leftrightarrow}(i, j, l)) \cdot \left\{ \sum_{k=0}^i \left(\int_0^l e^{-2\lambda \pi x^2 p^k} \lambda \theta x dx \right) \cdot p^k (1-p) \cdot g(k, j, l - \delta) \right. \\ &\quad \left. + \sum_{k=i+1}^{\infty} \left(\int_0^l e^{-2\lambda \pi x^2 p^{i+1}} \lambda \theta x dx \right) \cdot p^k (1-p) \cdot g(k, j, l - \delta) \right\} \\ &\geq g_{\leftrightarrow}(i, j, l) + (1 - g_{\leftrightarrow}(i, j, l)) \cdot \left\{ \frac{\theta}{4\pi} \cdot \sum_{k=0}^i \left(\frac{1 - e^{-2\pi l^2 p^k}}{p^k} \right) \cdot p^k (1-p) \cdot g(k, j, l - \delta) \right. \\ &\quad \left. + \frac{\theta}{4\pi} \cdot \left(\frac{1 - e^{-2\pi l^2 p^{i+1}}}{p^{i+1}} \right) \cdot \sum_{k=i+1}^{\infty} p^k (1-p) \cdot g(k, j, l - \delta) \right\}. \end{aligned}$$

We use the simple fact that for $k \leq i$

$$(1 - e^{-2\lambda \pi l^2 p^k}) \geq (1 - e^{-2\lambda \pi l^2 p^i}),$$

to get

$$\begin{aligned} g(i, j, l) &\geq g_{\leftrightarrow}(i, j, l) + (1 - g_{\leftrightarrow}(i, j, l)) \cdot \left\{ \frac{\theta}{4\pi} \cdot (1 - e^{-2\lambda \pi l^2 p^i}) \cdot \sum_{k=0}^i \left(\frac{1}{p^k} \right) \cdot p^k (1-p) \cdot g(k, j, l - \delta) \right. \\ &\quad \left. + \frac{\theta}{4\pi} \cdot \left(\frac{1 - e^{-2\lambda \pi l^2 p^{i+1}}}{p^{i+1}} \right) \cdot \sum_{k=i+1}^{\infty} p^k (1-p) \cdot g(k, j, l - \delta) \right\}. \end{aligned}$$

In order to simplify this we prove a small lemma:

Lemma 3.14 Given a parameter p s.t. $0 < p < 1$ and two sequences, $\{a_n = \frac{1}{p^n} : n \geq 0\}$ and $\{b_n : n \geq 0\}$ which takes non-negative values:

$$\sum_{n=0}^i a_n \cdot b_n \geq \frac{p^i(1-p)}{1-p^{i+1}} \cdot \left(\sum_{n=0}^i a_n \right) \cdot \left(\sum_{m=0}^i b_m \right).$$

Proof. We start from the right hand side of the inequality:

$$\begin{aligned} \sum_{n=0}^i a_n \cdot \sum_{n=0}^i b_n &= \sum_{n=0}^i a_n \cdot b_n + \sum_{n=0}^i b_n \cdot \sum_{m=0, m \neq n}^i a_m \\ &\leq \sum_{n=0}^i a_n \cdot b_n + \sum_{n=0}^i b_n \cdot \sum_{m=1}^i a_m \\ &= \sum_{n=0}^i a_n \cdot b_n + \sum_{n=0}^i b_n \cdot \sum_{m=1}^i \frac{1}{p^m} \\ &\leq \sum_{n=0}^i a_n \cdot b_n + \sum_{n=0}^i b_n \cdot \frac{1-p^i}{(1-p)p^i} \end{aligned}$$

Now, since $\{a_n\}$ is an increasing sequence whose minimum value is 1, achieved at $n = 0$ we can say that

$$\begin{aligned} \sum_{n=0}^i a_n \cdot \sum_{n=0}^i b_n &\leq \sum_{n=0}^i a_n \cdot b_n + \frac{1-p^i}{p^i(1-p)} \cdot \sum_{n=0}^i b_n \cdot a_n \\ &\leq \frac{1-p^i + p^i(1-p)}{p^i(1-p)} \cdot \sum_{n=0}^i b_n \cdot a_n. \end{aligned}$$

□

Using the inequality of Lemma 3.14 we get

$$\begin{aligned} g(i, j, l) &\geq g_{\leftrightarrow}(i, j, l) + \left\{ \frac{\theta}{4\pi} \cdot (1 - e^{-2\lambda\pi l^2 p^i}) \cdot \frac{p^i(1-p)}{1-p^{i+1}} \sum_{k=0}^i \left(\frac{1}{p^k} \right) \cdot \sum_{k=0}^i p^k(1-p) \cdot g(k, j, l - \delta) \right. \\ &\quad \left. + \frac{\theta}{4\pi} \cdot \left(\frac{1 - e^{-2\lambda\pi l^2 p^{i+1}}}{p^{i+1}} \right) \cdot \sum_{k=i+1}^{\infty} p^k(1-p) \cdot g(k, j, l - \delta) \right\} \cdot (1 - g_{\leftrightarrow}(i, j, l)) \\ &\geq g_{\leftrightarrow}(i, j, l) + \left\{ \frac{\theta}{4\pi} \cdot (1 - e^{-2\lambda\pi l^2 p^{i+1}}) \cdot \frac{1-p^{i+1}}{1-p^{i+1}} \cdot \sum_{k=0}^i p^k(1-p) \cdot g(k, j, l - \delta) \right. \\ &\quad \left. + \frac{\theta}{4\pi} \cdot \left(\frac{1 - e^{-2\lambda\pi l^2 p^{i+1}}}{p^{i+1}} \right) \cdot \sum_{k=i+1}^{\infty} p^k(1-p) \cdot g(k, j, l - \delta) \right\} \cdot (1 - g_{\leftrightarrow}(i, j, l)) \end{aligned}$$

We now get a form which is easier to handle i.e.

$$g(i, j, l) \geq g_{\leftrightarrow}(i, j, l) + (1 - g_{\leftrightarrow}(i, j, l)) \frac{\theta}{4\pi} \left(1 - e^{-2\lambda\pi l^2 p^{i+1}} \right) \cdot \sum_{k=0}^{\infty} p^k(1-p) g(k, j, l - \delta). \quad (3)$$

Observing that $g(l) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p^i(1-p)p^j(1-p)g(i, j, l)$, and noting that for $i \leq j$,

$$g_{\leftrightarrow}(i, j, l) = e^{-\lambda\pi l^2 p^{i+1}},$$

the inequality of (3) gives us

$$\begin{aligned} g(l) &\geq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p^i(1-p)p^j(1-p)g_{\leftrightarrow}(i, j, l) \\ &\quad + \sum_{i=0}^{\infty} p^i(1-p) \frac{\theta}{4\pi} \cdot \left(1 - e^{-2\lambda\pi l^2 p^{i+1}} \right) \cdot \sum_{j=0}^{i-1} \left(1 - e^{-\lambda\pi l^2 p^{j+1}} \right) \cdot \sum_{k=0}^{\infty} p^k(1-p)p^j(1-p) \cdot g(k, j, l - \delta) \\ &\quad + \sum_{i=0}^{\infty} p^i(1-p) \frac{\theta}{4\pi} \cdot \left(1 - e^{-2\lambda\pi l^2 p^{i+1}} \right) \cdot \left(1 - e^{-\lambda\pi l^2 p^{i+1}} \right) \cdot \sum_{j=i}^{\infty} \sum_{k=0}^{\infty} p^k(1-p)p^j(1-p) \cdot g(k, j, l - \delta) \end{aligned}$$

Observe that

$$g_{\leftrightarrow}(l) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p^i(1-p)p^j(1-p)g_{\leftrightarrow}(i, j, l).$$

With this and the simple fact that for $j < i$

$$(1 - e^{-\lambda\pi l^2 p^j}) > (1 - e^{-\lambda\pi l^2 p^i}),$$

we get

$$g(l) \geq g_{\leftrightarrow}(l) + \sum_{i=0}^{\infty} p^i (1-p) \frac{\theta}{4\pi} \cdot \left(1 - e^{-2\lambda\pi l^2 p^{i+1}}\right) \cdot \left(1 - e^{-\lambda\pi l^2 p^{i+1}}\right) \cdot g(l - \delta).$$

We begin by lower bounding $g_{\leftrightarrow}(l)$.

$$\begin{aligned} g_{\leftrightarrow}(l) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p^i p^j (1-p)^2 e^{-\lambda\pi l^2 p^{\min\{i,j\}+1}} \\ &= 2 \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} p^i p^j (1-p)^2 e^{-\lambda\pi l^2 p^{i+1}} - \sum_{i=0}^{\infty} p^{2i} (1-p)^2 e^{-\lambda\pi l^2 p^{i+1}} \\ &= (1-p^2) \cdot \left(\sum_{i=0}^{\infty} p^{2i} e^{-\lambda\pi l^2 p^{i+1}} \right) \end{aligned}$$

Now we lower bound the summation by integrating over the decreasing part of the function $p^{2x} e^{-\lambda\pi l^2 p^{x+1}}$, after which we get

$$g_{\leftrightarrow}(l) \geq (1-p^2) \cdot \left(\sum_{i=0}^{\infty} p^{2i} e^{-\lambda\pi l^2 p^{i+1}} \right) \geq \frac{(1-3e^{-2})(1-p^2)}{(\lambda\pi l^2 p)^2}. \quad (4)$$

We now turn to the coefficient of $g(l - \delta)$ and observe that,

$$\begin{aligned} \sum_{i=0}^{\infty} p^i \cdot \left(1 - e^{-2\lambda\pi l^2 p p^i}\right) \left(1 - e^{-\lambda\pi l^2 p p^i}\right) &\geq \int_0^{\infty} p^x \cdot \left(1 - e^{-2\lambda\pi l^2 p p^x}\right) \cdot \left(1 - e^{-\lambda\pi l^2 p p^x}\right) dx \\ &= \frac{1}{\ln(1/p)} \cdot \left(1 - \frac{1 - e^{-2\lambda\pi l^2 p}}{2\lambda\pi l^2 p} - \frac{1 - e^{-\lambda\pi l^2 p}}{\lambda\pi l^2 p} + \frac{1 - e^{-3\lambda\pi l^2 p}}{3\lambda\pi l^2 p}\right). \end{aligned}$$

Note that

$$\frac{1 - e^{-3\lambda\pi l^2 p}}{3\lambda\pi l^2 p} \geq \frac{1 - e^{-2\lambda\pi l^2 p}}{2\lambda\pi l^2 p}$$

which gives us the recursion

$$\boxed{g(l) \geq \frac{(1-3e^{-2})(1-p^2)}{(\lambda\pi l^2 p)^2} + \frac{\theta}{4\pi} \cdot \frac{1-p}{\ln(1/p)} \cdot \left(1 - \frac{1 - e^{-\lambda\pi l^2 p}}{\lambda\pi l^2 p}\right) \cdot g(l - \delta).} \quad (5)$$

Consider the function

$$\alpha(p) = \frac{(1-p)}{\ln(1/p)} \cdot \left(1 - \frac{1 - e^{-\lambda\pi l^2 p}}{\lambda\pi l^2 p}\right)$$

For $\alpha(p)$ to be greater than some constant $\eta > 0$ we require

$$\lambda\pi l^2 p \left(1 - \eta \cdot \frac{\ln(1/p)}{1-p}\right) \geq 1 - e^{-\lambda\pi l^2 p}. \quad (6)$$

Under the condition that

$$\eta < \frac{1-p}{\ln(1/p)}. \quad (7)$$

we get that (6) is satisfied whenever

$$\lambda\pi l^2 p \left(1 - \eta \cdot \frac{\ln(1/p)}{1-p}\right) \geq 1.$$

We denote the least value of l for which this inequality is true by $\gamma(\eta)$ i.e.

$$\gamma(\eta) = \sqrt{\frac{1}{\lambda\pi p \cdot \left(1 - \eta \cdot \frac{\ln(1/p)}{1-p}\right)}}. \quad (8)$$

Now, in order to lower bound the probability of u and v having a θ -good path, we choose an $\eta > 0$ satisfying (7) and unfold the recursion obtained in (5) till the path either connects directly to v or reaches a point w such that $d(w, v) \leq \gamma(\eta)$. Observing that $\frac{(1-3e^{-2})(1-p^2)}{(\lambda\pi l^2 p)^2} < \frac{(1-3e^{-2})(1-p^2)}{(\lambda\pi(l-\delta)^2 p)^2}$, and that the number of edges we may have in the path is finite but unbounded we get a geometric series sum when we open the recursion, giving us

$$g(l) \geq \frac{(1-3e^{-2})(1-p^2)}{(\lambda\pi l^2 p)^2} \cdot \frac{1}{1 - \frac{\eta\theta}{2\pi}} \cdot g(l^*), \quad (9)$$

where $l^* \leq \gamma(\eta)$. Let us assume that the point we have reached is w with $l^* = d(u, w) < \gamma(\eta)$. From this point we only consider the case that w connects directly to v . Using the lower bound of (4) we get that

$$g(l^*) \geq g_{\leftrightarrow}(l^*) \geq \frac{(1-3e^{-2})(1-p^2)}{(\lambda\pi l^{*2} p)^2} \geq \frac{(1-3e^{-2})(1-p^2)}{(\lambda\pi\gamma(\eta)^2 p)^2}.$$

Substituting this in (9) we get

$$g(l) \geq \frac{(1-3e^{-2})(1-p^2)}{(\lambda\pi\gamma(\eta)^2 p)^2} \cdot \frac{(1-3e^{-2})(1-p^2)}{(\lambda\pi l^2 p)^2} \cdot \frac{1}{1 - \frac{\eta\theta}{2\pi}}.$$

Substituting the value of $\gamma(\eta)$ from (8) we get

$$g(l) \geq \left(1 - \eta \cdot \frac{\ln(1/p)}{1-p}\right)^2 \cdot \frac{(1-3e^{-2})^2(1-p^2)^2}{(\lambda\pi l^2 p)^2} \cdot \frac{1}{1 - \frac{\eta\theta}{2\pi}}.$$

□

4 Hierarchical neighbor graphs as a clustering mechanism for WSNs

In this section we simulate the situation where a hierarchical neighbor graph architecture is used as a clustering mechanism to collect data from a field of ad hoc wireless sensor devices. We compare the performance of our architecture to that of LEACH [8] under assumptions similar to theirs and find that hierarchical neighbor graphs compare favorably.

The quality of a clustering architecture that collects data from wireless sensor networks is measured by determining: (a) Data throughput. In certain sensor networks applications, the amount of raw data that is communicated may not be as important as the information content, or the *effective data*, since the data from certain nodes may be redundant. Hence we consider effective data throughput as measure of network performance. (b) Energy efficiency i.e. the energy expended in transmitting a certain amount of data to the base station. (c) Network Lifetime. Often it is not possible to recharge node batteries. We therefore study the *network lifetime*, which we define here as the time it takes for all nodes in the network to die out.

We consider a typical field of wireless sensors with a set of nodes V in which a large number of sensor nodes continuously sense data, process it and communicate the information to an external sink or a base-station (BS). The sensors coordinate among themselves to form a hierarchical communication network, its architecture determined by $\text{HN}_p^w(V)$ where the weight function w depends on the battery power available at a node at any point of time. Nodes at higher levels of the hierarchy get depleted of their energy quicker, and so there is a heterogeneity in the power profile of the network. We periodically reform the network to distribute the energy load according to the residual energy at each node. Once the network topology has been formed, each node sensor constantly monitors its environment and periodically sends the data up the hierarchy to its parent. The nodes at the top-most level of $\text{HN}_p^w(V)$ communicate directly with the BS.

Sensor data is often highly correlated locally so we aggregate data, reducing the number of messages transmitted to the BS, hence improving the energy efficiency of the network. We consider two different applications of WSNs and the corresponding data aggregation models used. (a) *Limited Aggregation*: Only data signals from nodes located close to each other are highly correlated and can be aggregated into a single signal. We believe that in $\text{HN}_p^w(V)$, nodes which share a common parent are located close to each other and hence the data signals are correlated. In this case, a bounded number of data signals are fused into a single data signal. (b) *Unlimited Aggregation*: All data signals, irrespective of location can be fused to get a single signal. This model is valid for applications in which we are interested in quantities like the average, min or max of a set of values e.g. radiation level monitoring in a nuclear plant where the most useful information for the safety of plant is the maximum value [18]. In this case, all data signals at a relaying node are fused into a single signal.

Network setup and operation. We assume that each node has the computational ability to support MAC protocols and perform the signal processing functions required. To begin with, nodes V organize themselves into $\text{HN}_p^w(V)$. The weight function w is given by $\text{batt}_u/\text{batt}_{th}$, where batt_u is the residual battery of node u and batt_{th} is the threshold energy below which a node is declared dead. All nodes $\{u \text{ s.t. } lev(u) = 0\}$, periodically transmit data to their parents. All other nodes receive data from all their children, fuse the incoming data signals along with their own data according to the data aggregation model and transmit to their parent. The nodes at the topmost level transmit data to the BS. Nodes $\{u \text{ s.t. } lev(u) = 0\}$ switch off to conserve energy when they are not transmitting. The operation of the network is divided into rounds. The network is reformed after the end of each round, which is after a fixed duration. We assume collision free traffic in the network. This requires each node in $\text{HN}_p(V)$ to create a TDMA schedule and communicate this to their children. In addition the BS distributes spreading codes for direct-sequence spread spectrum (DSSS) [9] among the nodes in the networks.

Simulations and Discussion In this section we simulate $\text{HN}_p^w(V)$ and compare it to LEACH [8]. We compare the energy consumption, network lifetime and the effective data throughput received at the BS. The effective data throughput is measured by the number of data signals represented by the aggregated signal received at the BS. $\text{HN}_p^w(V)$ was observed to transmit 8 times more effective data than LEACH, when using unlimited data aggregation, and 2 to 3 times more when using limited data compression

Now we describe the simulation environment and parameters. We mimicked LEACH's setup in order to compare our structure with theirs [8]. We initialized a network with 100 sensor nodes, V , spread uniformly over a square region of side 100 unit, each having the same energy of $2J$ to start with. We considered a node to be dead if node energy drops below $0.1J$. We constructed $\text{HN}_p^w(V)$, with parameter $p = 0.5$, periodically at intervals of 20s. The BS is located close to A but outside it. The bandwidth of the channel was taken to be 1 Mb/s, and each data signal was taken to be 500 bytes long with a 25 byte packet header for each type of packet. We simulated this scenario and compared our mechanism against LEACH, observing energy consumption and throughput in the network. The time for a round was chosen to be 20s.

In LEACH for a $N = 100$ node network with $k = 5$ cluster heads, approximately $N/k = 20$ signals are fused into a single signal at each cluster head, assuming a uniform distribution. We simulated our setup for the same 20:1 compression ratio, as well as for a lower compression ratio of 10:1. In the second data aggregation model, each node fuses all signals present to a single signal and then transmits to its parent.

Nodes at $\{u \text{ s.t. } lev(u) = 0\}$ sense data and send to their respective parents. Nodes at level $i(> 1)$ receive data from all children, process the data and remove redundancy, and forward the effective data to their parent. This cycle continues till the highest level when data is sent to the BS. Nodes at the lowest level usually spend lesser amount of energy, whereas those at higher level will be quickly depleted of their battery power.

For our simulation we assume a simple model for energy dissipation (the same model as used to study LEACH in [8]). To transmit l bits of data over a distance d , the energy dissipated is $E_{Tx}(l, d) = lE_{elec} + l\epsilon_{fs}d^2$, and to receive a l -bit message is $E_{Rx}(l) = lE_{elec}$. The radio electronics energy $E_{elec} = 50 \text{ nJ/bit}$, depends on the coding and spreading of the signal. And the amplifier constant $\epsilon_{fs} = 10 \text{ pJ/bit/m}^2$ depends on the acceptable bit-error rate. In addition energy is consumed for data aggregation which is taken to be $E_{DA} = 5 \text{ nJ/bit/signal}$. We work on the assumption that the nodes are capable of controlling their power in order to vary the transmission radius.

Our simulation results are presented in Figures 7, 8, 9. The data points for LEACH are taken from [8]. Clearly when using unlimited data aggregation (" $\text{HN}_p^w(V)$ Unlimited" in the plots), it is observed that the throughput (Figure 7) and the ratio of data sent to energy consumed (Figure 8) are much higher than for LEACH, which is expected since our structure aggregates at several levels. Even when we limit the amount of aggregation allowed (in the plots " $\text{HN}_p^w(V)$ Limited < 20 " represents a 20:1 compression and " $\text{HN}_p^w(V)$ Limited < 10 " represent

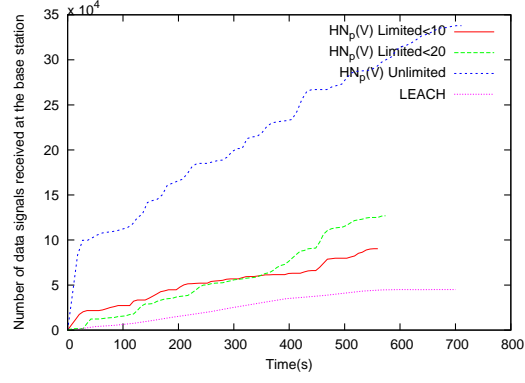


Figure 7: Total amount of data received at the BS over time.

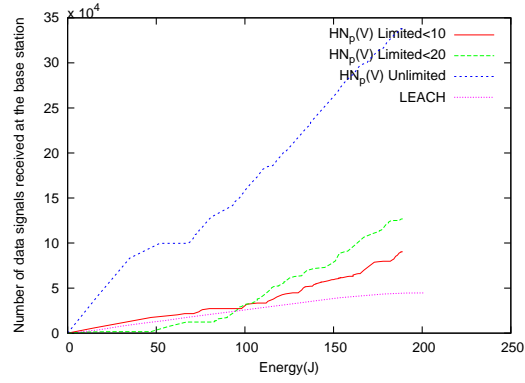


Figure 8: Total amount of data received at the BS per given amount of energy.

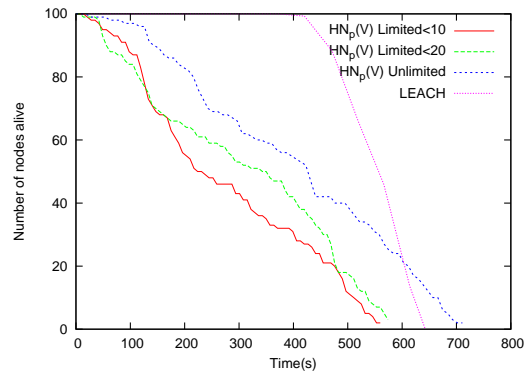


Figure 9: Number of nodes alive over time.

10:1) $\text{HN}_p^w(V)$ outperforms LEACH in these two aspects. With network lifetime the picture is a little more complex. In Figure 9 we see that in $\text{HN}_p^w(V)$ the time the last node dies is more or less the same as LEACH, although LEACH loses a lot of nodes suddenly while $\text{HN}_p^w(V)$ degrades slowly. In conclusion, it appears from the simulation that our structure does not dominate LEACH in terms of lifetime but uses energy more efficiently and provides a higher throughput than LEACH does, running far ahead in the case of highly compressible data.

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